1.3 Geometric Sequences and Series

Remember Sarah's story from 1.2? On her 8th birthday she was given $100 and then given the opportunity to earn $10 every week for doing work at home.

On her 9th birthday her parents encouraged Sarah to invest $600 of her money in a savings account. Her savings account earns 5% compound interest, compounded monthly. After 1 month, she earned 5% of $600, an additional $30.

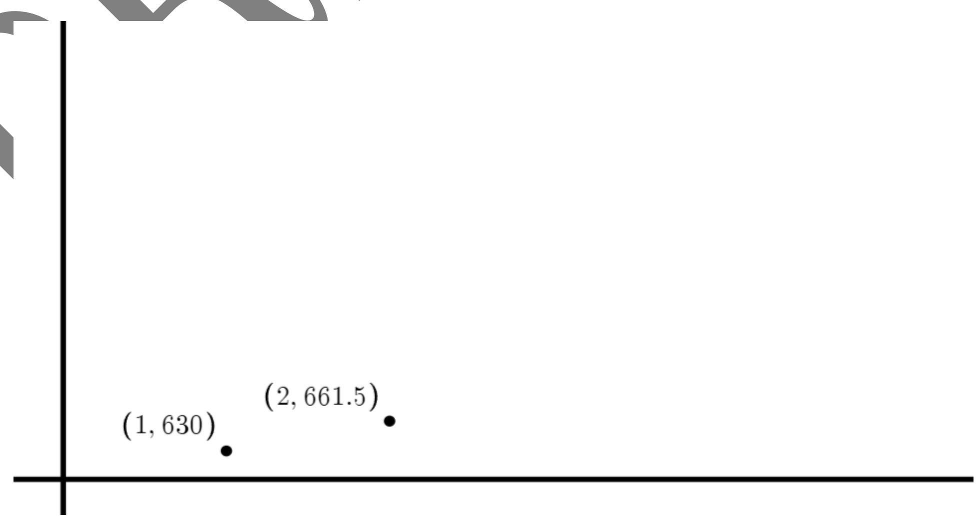
> **Quick Tip:** If you want to calculate the **growth** with a percentage, do it this way.
>
> $$600 \times 1.05 = 630$$
>
> Why times 1.05? Well, you have 100% of the 600, plus the 5%, which is 0.05.

With compound interest, the next month's interest includes the interest earned in the previous month(s). So, for the second month, Sarah earns 5% of $630, giving her $661.50.

Fill out the table below showing how much money she has at the end of each month.

Month	1	2	3	4	5	6
Total	$630	$661.50				

Now, on the graph below, plot those points. Be sure to label the *x* and *y* axes!



If you were to connect the points, what type of function would best model this trend?

Conceptual Connection: Geometric vs. Arithmetic

Arithmetic sequences add the same amount repeatedly. That's why they link to linear equations and why the formula looks like a straight-line pattern.

But geometric sequences? They multiply by the same number every time. That's a huge shift.

Think of it this way:

Repeated addition → multiplication
Repeated multiplication → exponents

So instead of something like:
3, 6, 9, 12, 15... (Add 3 each time — arithmetic)
You get:
3, 6, 12, 24, 48... (Multiply by 2 each time — geometric)

In arithmetic, the equation looks like:
$u_n = u_1 + (n - 1)d$ (linear)

In geometric, the growth is exponential:
$u_n = u_1 \times r^{n-1}$ (exponential)

This is why geometric patterns curve when graphed—they speed up (or slow down in decay) because each step changes more dramatically than the last.

Example 1:

The first term of a geometric sequence is 3, and the common ratio is 2.
Find the 10th term.

Step 1: Goal – Find the 10th term → u_{10}

Step 2: Clues – $u_1 = 3$, $r = 2$, $n = 10$
Step 3: Tools – Use the general term formula: $u_n = u_1 \times r^{n-1}$

Step 4: Plug in values – $u_{10} = 3 \cdot 2^9$

Step 5: Solve – 1536

Step 6: Verify – Does this make sense? The sequence grows quickly: 3, 6, 12, 24, … Yep!

Example 2:

A medicine dose is 100 mg. Each hour, the amount in the body decreases by 30%.
Find the amount remaining after 5 hours.

Step 1: Goal: Find u_6 (note: initial dose is u_1)
Step 2: Clues: $u_1 = 100$, r = 0.7 (since 30% is lost), n = 6
Step 3: Tool: Use $u_n = u_1 \times r^{n-1}$

Step 4: Plug in the values: $u_6 = 100 \times 0.7^5$

Step 5: Solve: $u_6 \approx 16.81$

Step 6: Verify: It is hard to verify this, but since we are raising a number less than 1 to a positive exponent, that's the same as repeated division. We would expect a smaller number. It seems plausible.

Geometric Series

We have already seen the difference between a sequence and a series in section 1.2 on arithmetic sequences and series. To recap, a series is the sum of the terms in a sequence. The sequence is the progression of the values.

The formula for a geometric series (sum) is shown below.

$$S_n = \frac{u_1\left(1-r^n\right)}{1-r} \ (\text{when } r \neq 1)$$

The formula booklet for IB Math also shows the formula this way:

$$S_n = \frac{u_1\left(r^n-1\right)}{r-1} \ (\text{when } r \neq 1)$$

Example 3:

You invest $1,000 in a crypto fund that doubles each year. What is the total value after 6 years?

Step 1: Goal: Find S_6, the total value

Step 2: Clues: $u_1 = 1000$, r = 2, n = 6

Step 3: Tool: Use $S_n = \frac{u_1\left(1-r^n\right)}{1-r}$

Step 4: Plug in: $S_6 = \frac{1000\left(1-2^6\right)}{1-2}$

Step 5: Solve/Evaluate:

$S_6 = 63{,}000$

Step 6: Verify: Wow! That's exponential growth in action.

You Try (Geometric Sequences)

1. The first term is 5 and r = 3. Find the 7th term.
2. A population of bacteria triples every hour. Starting with 200 bacteria, how many after 4 hours?
3. A tree's height is 10 feet and grows by 5% annually. Estimate height after 8 years.
4. A radioactive element halves every 10 years. Its current mass is 80g. What will remain after 30 years?
5. A sequence has $u_1 = 1200$ and $u_4 = 150$. Find the common ratio and u_6.

You Try (Geometric Series)

1. $u_1 = 3$, r = 2, n = 6. Find the sum.
2. A phone value starts at $900 and drops by 25% each year. Find total value lost after 3 years.
3. A student gets a scholarship starting at $1,000 that increases by 10% each semester. How much is earned in total over 4 semesters?
4. Find the sum of the first 8 terms of the sequence: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
5. A geometric series has $u_1 = 5$ and r = -2. Find S_5.

Answer Key

Geometric Sequences

1. $u_7 = 5 \times 3^6 = 5 \times 729 = 3645$

2. $u_5 = 200 \times 3^4 = 200 \times 81 = 16{,}200$

3. $u_9 = 10 \times 1.05^8 \approx 10 \times 1.477455 = 14.77$ ft

4. $u_4 = 80 \times 0.5^3 = 80 \times 0.125 = 10$ g

5. $u_4 = 1200 \times r^3 = 150 \rightarrow r^3 = 1/8 \rightarrow r = 1/2 \rightarrow u_6 = 1200 \times (1/2)^5 = 1200 \times 1/32 = 37.5$

Geometric Series

1. $S_6 = 3 \times (1 - 2^6)/(1 - 2) = 3 \times (1 - 64)/(-1) = 3 \times 63 = 189$

2. Final value = 900 × 0.75^3 ≈ 379.69 → Total loss = 900 - 379.69 = 520.31

3. S_4 = 1000 × (1 - 1.1^4)/(1 - 1.1) ≈ 1000 × (-0.4641)/(-0.1) = 4641

4. S_8 = (1 - $(1/2)^8$)/(1 - 1/2) = (255/256)/(1/2) = 255/128 = 1.992

5. S_5 = 5 × (1 - $(-2)^5$)/(1 - (-2)) = 5 × (1 + 32)/3 = 5 × 11 = 55

Sigma Notation:

Sigma notation is a shorthand way of writing long sums, especially when dealing with sequences and series. The Greek letter Σ (sigma) means "sum."

A typical example looks like this:

$$\sum_{n=1}^{5} u_n$$

This means:

"Add up the terms of the sequence u_n, starting at n = 1 and ending at n = 5."

Let's see another example.

$$\sum_{n=1}^{5} 2^n$$

This means we have to do two things. First, plug in the integer values for n starting at 1 (the number listed below sigma) and ending on 5 (the number on top of sigma). Second, add those together. The result is below.

$$\sum_{n=1}^{5} 2^n = 2^1 + 2^2 + 2^3 + 2^4 + 2^5$$

$$\sum_{n=1}^{5} 2^n = 2 + 4 + 8 + 16 + 32 = 62$$

$$\sum_{n=1}^{5} 2^n = 62$$

What does Sigma Notation have to do with Series?

Well, I'm so glad you asked! Since a series is a sum, sometimes you will be prompted to find the series with sum notation.

When this happens, you have two choices.

1. Add it up manually, like in the previous example, or
2. Use the geometric series formula or the arithmetic series formula

Example – Sigma Notation and Series

$$\sum_{n=3}^{8} 3 \cdot 2^{(n-1)}$$

Step 1: State the goal.

- We need to find the sum between **3 and 8**, for the equation $3 \cdot 2^{(n-1)}$.

Step 2: What are the clues?

- We know we start on 3 and end on 8, and the formula is $3 \cdot 2^{(n-1)}$, and this is **geometric**.
  - How do we know this is geometric, not arithmetic? Because the input (*n*) is an exponent!

Step 3: What tools do we have? A formula.

- $S_n = \frac{u_n\left(1-r^n\right)}{1-r}$
- What does *n* equal for 3 through 8? Remember, 3 is included, so we need the 3rd, 4th, 5th, 6th, 7th, and 8th terms, that's 6 terms! So, ***n* = 6**.
- In the formula we need *r* and u_1.
  - *r* is the base, 2.
  - Since we start on *n* = 3, we will find u_3.

$$u_3 = 3 \cdot 2^{(3-1)} \rightarrow u_3 = 12$$

Step 4: Plug everything in.

$$S_n = \frac{u_n\left(1-r^n\right)}{1-r}$$

$$S_6 = \frac{12\left(1-2^6\right)}{1-2}$$

Step 5: Simplify/solve/evaluate.

$$S_6 = \frac{12(1-64)}{-1} \rightarrow S_6 = 756$$

Example 2: Sigma Notation and Arithmetic Series

Evaluate the following.

$$\sum_{n=5}^{12}(5n+5)$$

You try this one. Figure out *n*, write the arithmetic series formula out, find the missing parts, plug everything in.

Did you get 380?

Practice Set A – Routine Practice

1. The first term of a geometric sequence is 7 and the common ratio is 4.
   Find the 6th term.
2. In a geometric sequence, $u_1 = 1250$, and $r = 0.8$. Find the 5th term.
3. A geometric sequence has $u_3 = 81$, and $r = 3$. Find the value of u_1.
4. Find the sum of the first 5 terms of the geometric series where the first term is 2 and $r = 5$.
5. A geometric series has $u_1 = 640, r = \frac{1}{2}$. Find the sum of the first 6 terms. Write your answer in standard form.
6. The 4th term of a geometric series is 54 and the common ratio is 3.
   Find the sum of the first 4 terms.

Practice Set B – IB-Style Questions (10 Total)

1. Leo starts a lawn mowing business during the summer. On his first day, he charges $40 per lawn and completes 2 lawns. Each day, he increases the number of lawns by 1 and raises his rate by $5 per lawn.

   By the 6th day, Leo is charging $65 per lawn.

   a. (i) Show that Leo charges $45 per lawn on the second day.
      (ii) Find the expression for the amount he charges per lawn on day *n*.
      (iii) Find the number of lawns he mows on day *n*. [5]

b. Let T_n represent the total earnings on day n. Show that
$T_n = \left(40+5(n-1)\right)\cdot\left(2+(n-1)\right)$.
Hence, find Leo's total earnings over the first 10 days. [4]

Leo invests all his earnings at the end of the 10th day into a digital portfolio that earns **3.2% compound interest monthly**.

c. (i) Write an expression to calculate the total value of his investment after 12 months.
(ii) Evaluate this expression to the nearest dollar.
(iii) Explain what the common ratio in this expression represents.[5]

d. Would it be more accurate to model Leo's monthly returns using a geometric series or a linear model? Justify your choice using features of each. [2]

2. Ava subscribes to a digital art platform. The subscription starts at $10 per month. Each year, the subscription cost increases by $2 per month. She keeps the subscription for 4 years.

a. Show that Ava pays $10 per month during the first year and $12 per month during the second year.
(ii) Write an expression for the monthly cost during year n.
(iii) Hence, calculate the total cost Ava pays over the 4 years. [5]

b. At the end of each year, Ava receives a cashback reward worth 5% of that year's total payments. These cashback amounts are invested and grow at 6% compounded annually.
i. (i) Calculate the cashback Ava receives at the end of each year.
(ii) Use sigma notation to write an expression for the total future value of her cashback investments after 4 more years.
(iii) Estimate the total value of Ava's investment to the nearest dollar.
[5]

c. Comment on whether Ava's payments form an arithmetic or geometric sequence. Does her investment growth model follow the same pattern? Explain. [2]

3. Mateo creates a workout plan. On Day 1, he does 10 push-ups. Each day, he does 5 more than the day before. On rest days, he skips push-ups entirely. Mateo works out for 30 days, resting every 7th day.

   a. Write an expression for the number of push-ups Mateo completes on workout day n.
      i. Determine how many rest days occur in the 30-day period.
      ii. Find the total number of push-ups Mateo completes in 30 days. [5]

   b. After the challenge, Mateo starts a new routine. He completes 50 push-ups on the first day, then increases the number by 10% each day for 14 days.
      i. Write a geometric expression for the total number of push-ups he completes during the 14-day period.
      ii. Use sigma notation to represent this total.
      iii. Evaluate the expression to the nearest whole number. [5]

   c. Compare the patterns in Mateo's original challenge and his new one. Which would be more effective for increasing strength over time? Justify your answer using mathematical reasoning. [2]

4. Nia launches a streaming channel. In the first week, she has 80 subscribers. The number of subscribers increases by 60 each week for the first 5 weeks. After that, her growth becomes exponential: her audience grows by 30% each week for the next 5 weeks.

   a. Write an expression for the number of subscribers in week n, for $1 \le n \le 5$.
      i. Find the total number of subscribers after 5 weeks. [3]

   b. Write a formula for the number of subscribers in week n, for $6 \le n \le 10$.
      i. Find the total number of subscribers in weeks 6 through 10. [4]

   c. Let $T = \sum_{n=1}^{10} S_n$, where S_n is the number of subscribers in week n.

      Calculate T, and explain why the shift from arithmetic to geometric growth significantly affects her channel. [3]

Answer Key

Practice Set A:

1. 7168
2. 48
3. 2.5
4. 1562
5. 3.75×10^{0}
6. 80

Practice Set B:

1. a i) Day 1: $40, Day 2, $5 increase - $45, a ii) $5n + 35$, a iii) $n + 1$, b) 4475, c i) $A = 4475\times(1.032)^{12}$, c ii) $6530, d) Geometric series is more accurate, as compound interest involves exponential growth due to interest earning interest, unlike linear's constant increase
2. a i) year 1: 10, year 2: 10 + 2 = 12, a ii) $C(n) = 10 + 2(n-1)$ or $C(n) = 2n + 8$, a iii) $624, b i) $6, $720, $8.40, $9.60, b ii) $\sum_{n=1}^{4} c_n (1.06)^{5-n}$, where $c_n =$ cash back in year n., b iii) ≈ $36, c) Each year her monthly subscription increases by $2, which is arithmetic (constant difference). Her investment is geometric because it is compound interest.
3. a i) $5n + 5$, a ii) 4 rest days, a iii) 1885, b i) $S_{14} = \frac{50(1-1.10^{14})}{1-1.10}$, b ii) $\sum_{n=0}^{13} 50\times1.10^{n}$, b iii) 1399 (pushups), c) Arithmetic is likely more effective for sustainable strength gains, as its gradual increase supports consistent progress without overtraining, unlike the geometric's rapid, potentially unsustainable escalation.
4. a) $S_n = 80 + (n-1)\times60$, or $S_n = 60n + 20$, a i) 1000, b) $320\times1.3^{n-6}$, b i) 2894, c) 3894